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A New Class of Long-Tailed Pausing Time Densities for the CTRW

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We present some asymptotic results for the family of pausing time densities having the asymptotic ($t \to \infty$) property $\psi(t) \sim [t \ln^{1+\gamma}(t/T)]^{-1}$. In particular, we show that for this class of pausing time densities the mean-squared displacement $\langle r^2(t) \rangle$ is asymptotically proportional to $\ln^{\gamma}(t/T)$, and the asymptotic distribution of the displacement has a negative exponential form.

KEY WORDS: Random walks; anomalous diffusion; disordered media.

A technique sometimes used to derive approximate results to describe transport in a disordered medium is that of the continuous-time random walk[(1)] (CTRW). The original idea of applying this methodology to such problems, first suggested by Scher and Lax,[(2)] was thereafter expanded in a number of further investigations, many of which are cited in ref. 3. Analyses of transport in a disordered or amorphous medium based on the CTRW can be regarded as a mean field theory since the pausing time density $\psi(t)$ is independent of the site. Thus, the question of whether it can deal with quenched disorder for any particular physical problem can only be settled by simulation studies. A common strategy used in CTRW studies of anomalous diffusion is to assume that the probability density for the pausing time $\psi(t)$ is such that the mean time between successive steps of the random walk is infinite. The most frequently used type of pausing time density is one having a stable law form, i.e., one which has the long-time behavior

$$\psi(t) \sim T^{\alpha}/t^{\alpha+1}, \qquad 0 < \alpha \leqslant 1 \tag{1}$$

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where T is a constant with the dimensions of time. The asymptotic form in this last equation suffices to produce anomalous diffusion in the sense that for a symmetric random walk the asymptotic form of the mean squared displacement goes like $\langle r^2(t)\rangle \sim (t/T)^\alpha$, where we have omitted an unimportant constant.

In this note we derive the analogous result for another class of long-tailed pausing time densities, whose behavior at long times is

$$\psi(t) \sim \frac{A}{t[\ln(t/T)]^{\gamma+1}}, \qquad \gamma > 0 \tag{2}$$

where again T is a constant with the dimensions of time and A is a dimensionless constant. One immediately evident distinction between the densities in Eqs. (1) and (2) is that in the first case moments of the waiting time of order less than α will be finite, while in the second there are no finite positive moments. Logarithmic moments of order less than γ will be finite for Eq. (2).

The Laplace transform of moments of the displacement of a random walker can be expressed in terms of the Laplace transform of $\psi(t)$, which we denote by $\hat{\psi}(s)$, and the asymptotic behavior of these moments can be expressed in terms of the behavior of this function for $s \to 0$. We therefore derive the small-s behavior of $\hat{\psi}(s)$ in order to calculate the asymptotic behavior of the mean-squared displacement for such CTRWs, finally comparing the results to those obtained by means of scaling arguments from an analysis of transport in a medium with quenched disorder.

Our analysis starts from the trivial identity $\hat{\psi}(s) = 1 - [1 - \hat{\psi}(s)]$, which allows us to focus on the s dependence of the term in brackets:

$$1 - \hat{\psi}(s) = \int_0^\infty [1 - e^{-st}]\, \psi(t)\, dt \tag{3}$$

A simple argument shows that the behavior of this function for $s \to 0$ can be found by substituting for $\psi(t)$ its asymptotic form [Eq. (2)], at the same time changing the lower limit of the integral to a nonzero value to avoid the apparent but not actual singularity at $t = 0$. This requires us to analyze the behavior of the integral

$$I(s) = A \int_{T_0}^\infty \frac{1 - e^{-st}}{t[\ln(t/T)]^{\gamma+1}}\, dt \tag{4}$$

as $s \to 0$. To do so, we differentiate the integral with respect to s, finding

$$I'(s) = A \int_{T_0}^{\infty} \frac{e^{-st}}{[\ln(t/T)]^{\gamma+1}} \, dt \tag{5}$$

At this point we can invoke an Abelian theorem for Laplace transforms[4] to infer that

$$I'(s) \sim \frac{A}{s[\ln(1/sT)]^{\gamma+1}} \tag{6}$$

as $s \to 0$, or

$$I(s) \sim A \int_0^s \frac{d\sigma}{\sigma[\ln(1/\sigma T)]^{\gamma+1}} = \frac{A}{T\gamma[\ln(1/sT)]^{\gamma}} \tag{7}$$

Hence, we conclude that, as $s \to 0$,

$$\hat{\psi}(s) \sim 1 - \frac{A'}{[\ln(1/(sT))]^{\gamma}} \tag{8}$$

where all of the constants have been lumped into the single A'.

We next examine the behavior of the mean-squared displacement of a symmetric CTRW when one assumes that the mean-square displacement of a single step of the underlying random walk is finite. If μ_2 is this moment, the Laplace transform of the variance $\langle \hat{r}^2(s) \rangle$ is given by

$$\langle \hat{r}^2(s) \rangle = \frac{\mu_2 \hat{\psi}(s)}{s[1 - \hat{\psi}(s)]} \sim \frac{\mu_2[\ln(1/sT)]^{\gamma}}{A's} \tag{9}$$

which, by a Tauberian theorem, implies that, aside from a multiplicative constant,

$$\langle r^2(t) \rangle \sim \ln^{\gamma}(t/T) \tag{10}$$

when $\psi(t)$ has the asymptotic behavior in Eq. (2).

In a recent paper scaling arguments have been used to analyze transport in a one-dimensional medium characterized by random transition rates and quenched disorder.[5,6] The probability density for any single transition rate was assumed to have the form

$$p(W) \sim \frac{1}{W[\ln(1/WT)]^{\gamma+1}} \tag{11}$$

Since the mean waiting time at any site is inversely proportional to the transition rate, the appropriate CTRW model is one in which $\psi(t)$ is that shown in Eq. (2). The results obtained in ref. 5 suggest that in one dimension

$$\langle x^2(t)\rangle \sim \left[\ln\left(\frac{t}{T}\right)\right]^{2\gamma}, \qquad t \gg T \tag{12}$$

while in two or more dimensions the relation (10) is valid. When $\psi(t)$ has an asymptotic stable law form, as in Eq. (1), one finds a similar discrepancy between results for a model with quenched disorder and the CTRW model as a function of the number of dimensions.[7] Results obtained for $\psi(t)$ with an asymptotic logarithmic tail [Eq. (2)] and those with the asymptotic stable law behavior of Eq. (1) are compared in Table I.

Table I. A Comparison of the Asymptotic Forms for the Mean Squared Displacements for CTRWs Having Pausing Time Densities in Eqs. (1) and (2), Respectively

	Quenched disorder	CTRW
$\psi(t) \sim T^\alpha/t^{\alpha+1}$	$D=1$: $\langle x^2\rangle \sim (t/T)^{2(1-\alpha)/(2-\alpha)}$ $D>1$: $\langle r^2\rangle \sim (t/T)^{1-\alpha}$	$\langle r^2\rangle \sim (t/T)^{1-\alpha}$
$\psi(t) \sim 1/\{t[\ln(t/T)]^{\gamma+1}\}$	$D=1$: $\langle x^2\rangle \sim \ln^{2\gamma}(t/T)$ $D>1$: $\langle r^2\rangle \sim \ln^{\gamma}(t/T)$	$\langle r^2\rangle \sim \ln^{\gamma}(t/T)$

It is also possible to find the asymptotic form for the probability density for the position of the random walker in one dimension $p(x, t)$, by starting from an integral representation of $\hat{p}(x, s)$:

$$\hat{p}(x, s) = \frac{1-\hat{\psi}(s)}{2\pi s}\int_{-\pi}^{\pi} \frac{\cos(x\theta)}{1-\hat{\psi}(s)\,\lambda(\theta)}\,d\theta \tag{13}$$

where $\lambda(\theta)$ is the structure function for the random walk. For simplicity of notation, let us write $\hat{\psi}(s) = 1-\varepsilon(s)$, where $\varepsilon(s) = -A' \ln^{-\gamma}(sT)$ in the limit $sT \to 0$. Let us further assume that the random walk is such that the mean squared displacement in a single step is finite, i.e., $\int_{-\infty}^{\infty} x^2 p(x)\,dx = \sigma^2 < \infty$. When this is the case we can find the large-$|x|$ limit of $p(x, t)$ by expanding $\lambda(\theta)$ around $\theta = 0$ as $\lambda(\theta) \sim 1 - \sigma^2\theta^2/2$ and taking the limits of integration to $\pm\infty$ because of the singularity at $\theta=0$ when s is set equal to 0. Thus, when the limit $sT\to 0$ is taken, $\hat{p}(x, s)$ can be approximated[8] by

$$\begin{aligned}\hat{p}(x, s) &= \frac{\varepsilon(s)}{2\pi s}\int_{-\pi}^{\pi}\frac{\cos(x\theta)}{1-[1-\varepsilon(s)]\,\lambda(\theta)}\,d\theta \\ &\sim \frac{\varepsilon(s)}{2\pi s}\int_{-\infty}^{\infty}\frac{\cos(x\theta)}{1-[1-\varepsilon(s)](1-\sigma^2\theta^2/2)}\,d\theta \\ &\sim \frac{\varepsilon(s)}{2\pi s}\int_{-\infty}^{\infty}\frac{\cos(x\theta)}{\varepsilon(s)+\sigma^2\theta^2/2}\,d\theta \\ &= \frac{1}{\sigma s}\left[\frac{\varepsilon(s)}{2}\right]^{1/2}\exp\left(-\frac{|x|\,[2\varepsilon(s)]^{1/2}}{\sigma}\right)\end{aligned}\tag{14}$$

Since $\varepsilon(s)$ in the present case is a slowly varying function,[4] we can invoke a Tauberian theorem for Laplace transforms[4] to infer that when $t\gg T$

$$p(x, t)\sim\frac{1}{\sigma}\left[\frac{A'}{\ln^{\gamma}(t/T)}\right]^{1/2}\exp\left[-\frac{(2A')^{1/2}}{\sigma}\frac{|x|}{\ln^{\gamma/2}(t/T)}\right]\tag{15}$$

which is similar to the form found by Kesten[9] for $p(x, t)$ in the Sinai model for diffusion in the presence of a particular form of random field.[10] Equation (15) is valid only in the tails of $p(x, t)$. In the neighborhood of $x=0$ one can follow the analysis of Weissman *et al.*[11] to show that $p(x, t)$ can be expanded as

$$p(x, t)\sim p(0, t)-x^2I(t)+O(x^4)\tag{16}$$

where

$$p(0, t)\sim\frac{1}{\sigma}\left[\frac{A'}{\ln^{\gamma}(t/T)}\right]^{1/2}\tag{17}$$

and

$$I(t)\sim\frac{A'}{4\pi\ln^{\gamma}(t/T)}\int_{-\pi}^{\pi}\frac{\theta^2}{1-\lambda(\theta)}\,d\theta\tag{18}$$

where the integral is a convergent one. A further result derivable in one dimension is that for the asymptotic survival probability for a random walker on a line of length L with traps located at $x=0$ and $x=L$. A formula for the Laplace transform can be found from a result for the survival probability in discrete time on a line given by Weiss and

Havlin.[12] When translated into continuous time, the formula for $\hat{S}(s)$ reads

$$\hat{S}(s) = \frac{4[1-\hat{\psi}(s)]}{\pi^2 s} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2[1-\hat{\psi}(s)+\pi^2\hat{\psi}(s)\,(2j+1)^2/L^2]} \tag{19}$$

On substituting the small-s form for $\hat{\psi}(s)$, we infer that the asymptotic survival probability goes like

$$S(t) \sim \frac{A'L^2}{24 \ln^{\gamma}(t/T)} \tag{20}$$

In three or more dimensions the general form for $p(\mathbf{r}, t)$ at large r will also have a scaling form similar to Eq. (15). For example, consider the case of a spherically symmetric random walk in three dimensions in the spatial regime in which x^2, y^2, and $z^2 \to \infty$. The asymptotic form of the propagator in this regime can be found by considering the properties of the integral representation of the Laplace transform of the propagator $\hat{p}(r, s)$,

$$\hat{p}(r, s) = \frac{1-\hat{\psi}(s)}{(2\pi)^3 s\hat{\psi}(s)} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\exp(i\mathbf{r}\cdot\boldsymbol{\theta})}{1/\hat{\psi}(s)-\lambda(\theta)}\, d^3\boldsymbol{\theta} \tag{21}$$

in the neighborhood of $\boldsymbol{\theta} = \mathbf{0}$. If we assume that $\lambda(\boldsymbol{\theta})$ can be expanded near the origin as $\lambda(\boldsymbol{\theta}) \sim 3 - \sigma^2\theta^2/2 + \cdots$, then Eq. (21) can be approximated by

$$\begin{aligned} \hat{p}(r, s) &\sim \frac{1-\hat{\psi}(s)}{(2\pi)^3 s\hat{\psi}(s)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\cos(x\theta_1)\cos(y\theta_2)\cos(z\theta_3)}{[1-\hat{\psi}(s)]/\hat{\psi}(s)+\sigma^2\theta^2/2}\, d^3\boldsymbol{\theta} \\ &\sim \frac{1-\hat{\psi}(s)}{2\pi\sigma^2 s\hat{\psi}(s) r} \exp\left[-\frac{r}{\sigma}\left(6\,\frac{1-\hat{\psi}(s)}{\hat{\psi}(s)}\right)^{1/2}\right] \end{aligned} \tag{22}$$

On making the expansion in Eq. (8), one finds that for small s

$$\hat{p}(r, s) \sim \frac{A'}{2\pi\sigma^2 rs \ln^{\gamma}(1/sT)} \exp\left[-\frac{r}{\sigma}\left(\frac{6A'}{\ln^{\gamma}(1/sT)}\right)^{1/2}\right] \tag{23}$$

which, following our earlier analysis, implies that

$$p(r, t) \sim \frac{A'}{2\pi\sigma^2 r \ln^{\gamma}(t/T)} \exp\left[-\frac{r}{\sigma}\left(\frac{6A'}{\ln^{\gamma}(t/T)}\right)^{1/2}\right] \tag{24}$$

for $t \to \infty$. To find the asymptotic time dependence of $p(0, t)$, we can use a similar argument to show that this function is inversely proportional to $\ln^{\gamma}(t/T)$ in the long-time limit in three dimensions.

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